

The chromatic spectrum of signed graphs

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Abstract

The chromatic number $\chi((G, \sigma))$ of a signed graph (G, σ) is the smallest number k for which there is a function $c : V(G) \rightarrow \mathbb{Z}_k$ such that $c(v) \neq \sigma(e)c(w)$ for every edge $e = vw$. Let $\Sigma(G)$ be the set of all signatures of G . We study the chromatic spectrum $\Sigma_\chi(G) = \{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$ of (G, σ) . Let $M_\chi(G) = \max\{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$, and $m_\chi(G) = \min\{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$. We show that $\Sigma_\chi(G) = \{k : m_\chi(G) \leq k \leq M_\chi(G)\}$. We also prove some basic facts for critical graphs.

Analogous results are obtained for a notion of vertex-coloring of signed graphs which was introduced by Máčajová, Raspaud, and Škovič in [2].

1 Introduction

Graphs in this paper are simple and finite. The vertex set of a graph G is denoted by $V(G)$, and the edge set by $E(G)$. A signed graph (G, σ) is a graph G and a function $\sigma : E(G) \rightarrow \{\pm 1\}$, which is called a signature of G . The set $N_\sigma = \{e : \sigma(e) = -1\}$ is the set of negative edges of (G, σ) and $E(G) - N_\sigma$ the set of positive edges. For $v \in V(G)$, let $E(v)$ be the set of edges which are incident to v . A switching at v defines a graph (G, σ') with $\sigma'(e) = -\sigma(e)$ for $e \in E(v)$ and $\sigma'(e) = \sigma(e)$ otherwise. Two signed graphs (G, σ) and (G, σ^*) are *equivalent* if they can be obtained from each other by a sequence of switchings. We also say that σ and σ^* are equivalent signatures of G .

A circuit in (G, σ) is balanced, if it contains an even number of negative edges; otherwise it is unbalanced. The graph (G, σ) is unbalanced, if it contains an unbalanced circuit; otherwise (G, σ) is balanced. It is well known (see e.g. [3]) that (G, σ) is balanced if and only if it is equivalent to the signed graph with no negative edges, and (G, σ) is antibalanced if it is equivalent to the signed graph with no positive edges. Note, that a balanced bipartite graph is also antibalanced. The underlying unsigned graph of (G, σ) is denoted by G .

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In the 1980s Zaslavsky [5, 6, 7] started studying vertex colorings of signed graphs. The natural constraints for a coloring c of a signed graph (G, σ) are, that (1) $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$, and (2) that the colors can be inverted under switching, i.e. equivalent signed graphs have the same chromatic number. In order to guarantee these properties of a coloring, Zaslavsky [5] used the set $\{-k, \dots, 0, \dots, k\}$ of $2k + 1$ "signed colors" and studied the interplay between colorings and zero-free colorings through the chromatic polynomial.

Recently, Máčajová, Raspaud, and Škoviera [2] modified this approach. If $n = 2k + 1$, then let $M_n = \{0, \pm 1, \dots, \pm k\}$, and if $n = 2k$, then let $M_n = \{\pm 1, \dots, \pm k\}$. A mapping c from $V(G)$ to M_n is a *signed n -coloring* of (G, σ) , if $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$. They define $\chi_{\pm}((G, \sigma))$ to be the smallest number n such that (G, σ) has a signed n -coloring. We also say that (G, σ) is *signed n -chromatic*.

In [1] we study circular coloring of signed graphs. The related integer k -coloring of a signed graph (G, σ) is defined as follows. Let \mathbb{Z}_k denote the cyclic group of integers modulo k , and the inverse of an element x is denoted by $-x$. A function $c : V(G) \rightarrow \mathbb{Z}_k$ is a k -coloring of (G, σ) if $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$. Clearly, such colorings satisfy the constraints (1) and (2) of a vertex coloring of signed graphs. The *chromatic number* of a signed graph (G, σ) is the smallest k such that (G, σ) has a k -coloring. We also say that (G, σ) is *k -chromatic*.

The following proposition describes the relation between these two coloring parameters for signed graphs.

Proposition 1.1 ([1]). *If (G, σ) is a signed graph, then $\chi_{\pm}((G, \sigma)) - 1 \leq \chi((G, \sigma)) \leq \chi_{\pm}((G, \sigma)) + 1$.*

Let G be a graph and $\Sigma(G)$ be the set of pairwise non-equivalent signatures on G .

The *chromatic spectrum* of G is the set $\{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$, which is denoted by $\Sigma_{\chi}(G)$. Analogously, *signed chromatic spectrum* of G is the set $\{\chi_{\pm}((G, \sigma)) : \sigma \in \Sigma(G)\}$. It is denoted by $\Sigma_{\chi_{\pm}}(G)$. Let $M_{\chi}(G) = \max\{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$ and $m_{\chi}(G) = \min\{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$. Analogously, $M_{\chi_{\pm}}(G) = \max\{\chi_{\pm}((G, \sigma)) : \sigma \in \Sigma(G)\}$ and $m_{\chi_{\pm}}(G) = \min\{\chi_{\pm}((G, \sigma)) : \sigma \in \Sigma(G)\}$.

The following theorems are our main results.

Theorem 1.2. *If G is a graph, then $\Sigma_{\chi}(G) = \{k : m_{\chi}(G) \leq k \leq M_{\chi}(G)\}$.*

Theorem 1.3. *If G is a graph, then $\Sigma_{\chi_{\pm}}(G) = \{k : m_{\chi_{\pm}}(G) \leq k \leq M_{\chi_{\pm}}(G)\}$.*

Theorems 1.2 and 1.3 will be proved in Sections 2 and 3, respectively.

2 The chromatic spectrum of a graph

We start with the determination of $m_{\chi}(G)$.

Proposition 2.1. *Let G be a nonempty graph. The following statements hold.*

1. $\Sigma_\chi(G) = \{1\}$ if and only if $m_\chi(G) = 1$ if and only if $E(G) = \emptyset$.
2. if $E(G) \neq \emptyset$, then $\Sigma_\chi(G) = \{2\}$ if and only if $m_\chi(G) = 2$ if and only if G is bipartite.
3. If G is not bipartite, then $m_\chi(G) = 3$.

Proof. Statements 1. and 2. are obvious. For statement 3 consider (G, σ) where σ is the signature with all edges negative. Then $c : V(G) \rightarrow \mathbb{Z}_3$ with $c(v) = 1$ is a 3-coloring of G . Since G is not bipartite the statement follows with statements 1. and 2. \square

If (G, σ) is a signed graph and $u \in V(G)$, then σ_u denotes the restriction of σ to $G - u$. A k -chromatic graph (G, σ) is *k -chromatic critical* if $\chi((G - u, \sigma_u)) < k$, for every $u \in V(G)$. In the following proposition, we will give some basic facts on k -chromatic critical graphs. The complete graph on n vertices is denoted by K_n .

Proposition 2.2. *Let (G, σ) be a signed graph.*

1. (G, σ) is 1-critical if and only if $G = K_1$
2. (G, σ) is 2-critical if and only if $G = K_2$.
3. (G, σ) is 3-critical if and only if G is an odd circuit.

Proof. Statements 1. and 2. are obvious. An odd circuit with any signature is 3-critical. For the other direction let G be a 3-critical graph. Note, that (*) $G - u$ is bipartite for every $u \in V(G)$ by Lemma 2.1. Since G is not bipartite it follows that every vertex of G is contained in all odd circuits of G , and by (*) every odd circuit C is hamiltonian. C cannot contain a chord, since for otherwise G contains a non-hamiltonian odd circuit, a contradiction. Hence, G is an odd circuit. \square

Lemma 2.3. *Let $k \geq 1$ be an integer. If (G, σ) is k -chromatic, then $\chi((G - u, \sigma_u)) \in \{k, k - 1\}$, for every $u \in V(G)$. In particular, if (G, σ) is k -critical, then $\chi((G - u, \sigma_u)) = k - 1$.*

Proof. For $k \in \{1, 2\}$, the statement follows with Proposition 2.1. Hence, we may assume that $k \geq 3$. Clearly, $\chi((G - u, \sigma_u)) \leq \chi((G, \sigma)) = k$. Suppose to the contrary that $\chi((G - u, \sigma_u)) \leq k - 2$, and let ϕ be a $(k - 2)$ -coloring of $(G - u, \sigma_u)$. We extend ϕ to a $(k - 1)$ -coloring of (G, σ) . If k is odd, then change color x to $x + 1$ for each $x \geq \frac{k-1}{2}$ and assign color $\frac{k-1}{2}$ to vertex u , and we are done. If k is even, then change color x to $x + 1$ for each $x \geq \frac{k}{2}$, and assign color $\frac{k}{2}$ to vertex u . If $\phi(v) = \frac{k-2}{2}$ for a vertex v and $\sigma(uv) = -1$, then recolor v with color $\frac{k}{2}$ to obtain a $(k - 1)$ -coloring of (G, σ) . Hence $\chi((G, \sigma)) \leq k - 1 < k$, a contradiction. Clearly, if (G, σ) is k -critical, then $\chi((G - u, \sigma_u)) = k - 1$. \square

The following statements are direct consequence of Lemma 2.3.

Theorem 2.4. *Let (G, σ) be a signed graph and $k \geq 1$. If $\chi((G, \sigma)) = k$, then (G, σ) contains an induced i -critical subgraph for each $i \in \{1, \dots, k\}$.*

Lemma 2.5. *Let $k \geq 3$ be an integer and H be an induced subgraph of a graph G . If $k \in \Sigma_\chi(H)$, then $k \in \Sigma_\chi(G)$.*

Proof. If $k \in \Sigma_\chi(H)$, then there is a signature σ of H such that $\chi((H, \sigma)) = k$. Let ϕ be a k -coloring of (H, σ) . Define a signature σ' of G as follows. Let $e \in E(G)$ with $e = uv$.

If $e \in E(H)$, then $\sigma'(e) = \sigma(e)$,

if $u, v \notin V(H)$ or if $u \in V(H), v \notin V(H)$ and $\phi(u) = 1$, then $\sigma'(e) = -1$,

if $u \in V(H), v \notin V(H)$ and $\phi(u) \neq 1$, then $\sigma'(e) = 1$.

It follows that ϕ can be extended to a k -coloring of (G, σ') by assigning color 1 to each vertex of $V(G) \setminus V(H)$. Thus $\chi((G, \sigma')) \leq k$. Moreover, (G, σ') has (H, σ) as a subgraph with chromatic number k , hence, $\chi((G, \sigma')) \geq k$. Therefore, $\chi((G, \sigma')) = k$ and thus, $k \in \Sigma_\chi(G)$. \square

Theorem 2.6. *Let $k \geq 4$ be an integer and G be a graph. If $k \in \Sigma_\chi(G)$, then $k-1 \in \Sigma_\chi(G)$.*

Proof. By Theorem 2.4, (G, σ) contains an induced k -critical subgraph (H, σ') , where σ' is the restriction of σ to H . Since $k \geq 4$, it follows that $|V(H)| > 3$. Hence, there is $u \in V(H)$ such that $\chi(H - u, \sigma'_u) = k - 1$. Furthermore, $H - u$ is an induced subgraph of G . Thus, $k - 1 \in \Sigma_\chi(H - u)$, and hence, $k - 1 \in \Sigma_\chi(G)$ by Lemma 2.5.

Note that if $k = 3$, then by Proposition 2.1, G is not a bipartite graph and thus k can not be decreased to 2. \square

Theorem 1.2 follows from Proposition 2.1 and Theorem 2.6.

3 The signed chromatic spectrum of a graph

Proposition 3.1. *Let G be a nonempty graph. The following statements hold.*

1. $\Sigma_{\chi_\pm}(G) = \{1\}$ if and only if $E(G) = \emptyset$.

2. if $E(G) \neq \emptyset$, then $m_{\chi_\pm}(G) = 2$.

Proof. Statement 1. is obvious. For statement 2, since G has at least one edge it can not be colored by using only one color, hence $m_{\chi_\pm}(G) = 2$. \square

A signed k -chromatic graph (G, σ) is *signed k -chromatic critical* if $\chi_\pm((G - u, \sigma_u)) < k$, for every $u \in V(G)$. In [4] Schweser and Stiebitz defined a graph (G, σ) to be critical with

respect to χ_{\pm} if $\chi_{\pm}((H, \sigma')) < \chi_{\pm}((G, \sigma))$ for every proper signed subgraph (H, σ') of (G, σ) , where σ' is the restriction of σ to $E(H)$. However, for trees and circuits the two definitions coincide. The analog statement to Proposition 2.2 for signed colorings is due to Schweser and Stiebitz in [4].

Proposition 3.2 ([4]). *Let (G, σ) be a signed graph.*

1. *(G, σ) is signed 1-critical if and only if $G = K_1$*
2. *(G, σ) is signed 2-critical if and only if $G = K_2$.*
3. *(G, σ) is signed 3-critical if and only if G is a balanced odd circuit or an unbalanced even circuit.*

Lemma 3.3. *Let $k \geq 1$ be an integer. If (G, σ) is signed k -chromatic, then $\chi_{\pm}((G - u, \sigma_u)) \in \{k, k - 1\}$, for every $u \in V(G)$. In particular, if (G, σ) is signed k -critical, then $\chi_{\pm}((G - u, \sigma_u)) = k - 1$.*

Proof. For $k \in \{1, 2\}$, the statement follows with Proposition 3.1. Hence, we may assume that $k \geq 3$. Clearly, $\chi_{\pm}((G - u, \sigma_u)) \leq \chi_{\pm}((G, \sigma)) = k$. Suppose to the contrary that $\chi_{\pm}((G - u, \sigma_u)) \leq k - 2$ and let ϕ be a $(k - 2)$ -coloring of $(G - u, \sigma_u)$. We shall extend ϕ to a $(k - 1)$ -coloring of (G, σ) . If k is even, then assign color 0 to vertex u , we are done. If k is odd, then assign color $\frac{k-1}{2}$ to vertex u , and for each vertex v such that $\phi(v) = 0$ and $\sigma(uv) = -1$, recolor v with color $\frac{k-1}{2}$, and for each vertex v such that $\phi(v) = 0$ and $\sigma(uv) = 1$, recolor v with color $-\frac{k-1}{2}$ to obtain a $(k - 1)$ -coloring of (G, σ) . Hence $\chi_{\pm}((G, \sigma)) \leq k - 1 < k$, a contradiction. Clearly, if (G, σ) is signed k -critical, then $\chi_{\pm}((G - u, \sigma_u)) = k - 1$. \square

Theorem 3.4. *Let (G, σ) be a signed graph and $k \geq 1$. If $\chi_{\pm}((G, \sigma)) = k$, then (G, σ) contains an induced signed i -critical subgraph for each $i \in \{1, \dots, k\}$.*

Lemma 3.5. *Let $k \geq 2$ be an integer and H be an induced subgraph of a graph G . If $k \in \Sigma_{\chi_{\pm}}(H)$, then $k \in \Sigma_{\chi_{\pm}}(G)$.*

The proof is similar to the proof of Lemma 2.5.

Theorem 3.6. *Let $k \geq 3$ be an integer and G be a graph. If $k \in \Sigma_{\chi_{\pm}}(G)$, then $k - 1 \in \Sigma_{\chi_{\pm}}(G)$.*

Proof. By Theorem 3.4, (G, σ) contains an induced signed k -critical subgraph (H, σ') , where σ' is the restriction of σ to H . Since $k \geq 3$, it follows that $|V(H)| \geq 3$. Hence, there is $u \in V(H)$ such that $\chi_{\pm}(H - u, \sigma'_u) = k - 1$. Furthermore, $H - u$ is an induced subgraph of G . Thus, $k - 1 \in \Sigma_{\chi_{\pm}}(H - u)$, and hence, $k - 1 \in \Sigma_{\chi_{\pm}}(G)$ by Lemma 3.5. \square

Theorem 1.3 follows from Proposition 3.1 and Theorem 3.6.

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